Quantum Mechanics and a Completely Integrable Dynamical System

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Z. Naturforsch. **42 a**, 819–824 (1987); received April 7, 1987

From the eigenvalue equation $(H_0 + \lambda V)\,|\,\psi_n(\lambda)\rangle = E_n(\lambda)\,|\,\psi_n(\lambda)\rangle$ one can derive an autonomous system of first order differential equations for the eigenvalues $E_n(\lambda)$ and the matrix elements $V_{mn}(\lambda) = \langle\psi_m(\lambda)\,|\,V\,|\,\psi_n(\lambda)$, where λ is the independent variable. If the initial values $E_n(\lambda = 0)$ and $\psi_n(\lambda = 0)$ are known the differential equations can be solved. Thus one finds the "motion" of the energy levels $E_n(\lambda)$. Here we give two applications of this technique. Furthermore we describe the connection with the stationary state perturbation theory. We also derive the equations of motion for the extended case $H = H_0 + \lambda_1 V_1 + \lambda_2 V_2$. Finally we investigate the case where the Hamiltonian is given by a finite dimensional symmetric matrix and derive the energy dependent constants of motion. Several open questions are also discussed.

A basic problem in quantum mechanics and quantum field theory is the calculation of the spectrum for a given Hamiltonian operator. The Hamiltonian operator models the system under consideration. Then from the knowledge of the spectrum we can calculate other physical quantities such as free energy, entropy etc. In general it is assumed that the Hamiltonian operator acts in a Hilbert space $\mathscr{H}$. The exactly solvable models are very rare. Examples are the harmonic and displaced harmonic oscillator. More complicated examples are the quantum nonlinear Schrödinger equation and the XY-model in one dimension. Both can be solved with the help of the quantum inverse scattering approach. The equations of motion can be written in a Lax representation.

In almost all cases we are not able to calculate exactly the spectrum. Thus we are forced to determine approximatively to spectrum. In general two approaches are available. Since the (explicitly time-independent) Hamiltonian operator H acts in a Hilbert space $\mathscr{H}$ we use the fact that a Hilbert space $\mathscr{H}$ admits at least one basis, say $|\Phi_n\rangle$, $n \in I$ (I index set). It is assumed that the index set is countable. Then the matrix representation $H_{mn} = \langle\Phi_m\,|\,H\,|\,\Phi_n\rangle$ of the Hamiltonian operator H is calculated. If the Hamiltonian operator H admits symmetries then the underlying Hilbert space $\mathscr{H}$ has to be decomposed into invariant subspaces. From the matrix representation we calculate in general with the help of a computer program the spectrum.

In many cases we have to deal with an infinite matrix (for example if Bose operators enter the Hamiltonian operator). Then we have to truncate the infinite matrix in order to perform a computer calculation. However, even if we have a finite dimensional system such as spin-$\frac{1}{2}$ systems with N lattice points the matrix must be truncate if N is large since $\dim \mathscr{H} = 2^N$.

In the second approach it is assumed that the Hamiltonian operator H can be written in the form $H_\lambda = H_0 + \lambda V$ where H_0 is the unperturbed Hamiltonian operator, V is the perturbation (or interaction) and λ is the real coupling constant. Then we can apply stationary state perturbation theory. Here it is assumed that the hermitian operators H_0 and V are time-independent. Furthermore it is assumed that the spectra of H_0 and H_2 are discrete and bounded from below. If the Hamiltonian H_λ admits symmetries then the underlying Hilbert space has to be decomposed into invariant subspaces so that the eigenvalues are non-degenerate in these subspaces. It is assumed that there are no accidential degeneries. Let $E_n(0)$ and $|\psi_n(0)\rangle$ be the energy eigenvalues and (normalized) eigenfunctions of the unperturbed Hamiltonian operator H_0, respectively. Since we have assumed the eigenvalues of H_0 are non-degenerate we find in first order approximation of perturbation theory

$$E_n(\lambda) \approx E_n(0) + \lambda\,\langle\psi_n(0)\,|\,V\,|\,\psi_n(0)\rangle, \qquad (1)$$

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where $\langle | \rangle$ denotes the scalar product in the underlying Hilbert space. In second order of approximation we obtain

$$E_n(\lambda) \approx E_n(0) + \lambda \langle \psi_n(0) | V | \psi_n(0) \rangle + \lambda^2 \sum_{m(\neq n)} \frac{|V_{mn}|^2}{E_n(0) - E_m(0)}. \quad (2)$$

Recently Pechukas [1] and Yukawa [2, 3] discussed the "motion of energy levels $E_n(\lambda)$" where λ plays the rôle of the time. Using the orthogonality relation $\langle \psi_n(\lambda) | \psi_m(\lambda) \rangle = \delta_{nm}$, the completeness relation $1 = \Sigma_n |\psi_n(\lambda)\rangle \langle \psi_n(\lambda)|$ and the assumptions described above, these authors derived the following autonomous system of first order ordinary differential equations:

$$\frac{dE_n}{d\lambda} = p_n, \quad \frac{dp_n}{d\lambda} = 2 \sum_{m(\neq n)} \frac{V_{mn} V_{nm}}{E_n - E_m}, \quad (3)$$

$$\frac{dV_{mn}}{d\lambda} = \sum_{k(\neq m,n)} \left[V_{mk} V_{kn} \left(\frac{1}{E_m - E_k} + \frac{1}{E_n - E_k} \right) \right] - \frac{V_{mn}(p_m - p_n)}{E_m - E_n} \quad (m \neq n),$$

where

$$p_n(\lambda) := \langle \psi_n(\lambda) | V | \psi_n(\lambda) \rangle \quad \text{and}$$
$$V_{mn}(\lambda) := \langle \psi_m(\lambda) | V | \psi_n(\lambda) \rangle \quad (n \neq m)$$

with $V_{nm} = V_{mn}$. Here we have also assumed that the eigenfunctions are real orthonormal.

Pechukas [1] and Yukawa [2, 3] discussed the dynamical system (3) in connection with quantum chaos (compare [4] and references therein). Moreover, Yukawa [3] showed that the system (3) admits a Lax representation and is completely integrable.

The purpose of the present paper is fourfold. First of all we give two applications. Secondly we describe the connection with the stationary state perturbation theory. Then we consider the extended case $H = H_0 + \lambda_1 V_1 + \lambda_2 V_2$. Finally we investigate the case where the Hamiltonian H is given by a finite dimensional symmetric matrix and derive the energy dependent constants of motion. Several open questions are also discussed.

In our first example the (scaled) Hamiltonian operator H is given by the matrix representation

$$H = \begin{pmatrix} 1 & \lambda & 0 & 0 \\ \lambda & 2 & 2\lambda & 0 \\ 0 & 2\lambda & 3 & \lambda \\ 0 & 0 & \lambda & 4 \end{pmatrix}, \quad (4)$$

where λ is a real parameter ($0 \leqq \lambda < \infty$). The operator H does not admit any symmetry. We put $H_0 = \text{diag}(1, 2, 3, 4)$ where diag(...) denotes that H_0 is a diagonal matrix. Obviously the eigenvalues of H_0 are given by $E_0 = 1$, $E_1 = 2$, $E_2 = 3$ and $E_3 = 4$. The eigenfunctions are the standard basis in $\mathbb{R}^4$, namely $|\psi_0(0)\rangle = (1, 0, 0, 0)^T, \ldots, |\psi_3(0)\rangle = (0, 0, 0, 1)^T$ (T means transpose). In order to solve the dynamical system (3) where $N = 4$ we have to determine the initial conditions. Since

$$V = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad (5)$$

we obtain $p_0(0) = \langle \psi_0(0) | V | \psi_0(0) \rangle = 0$. Analogously, we find $p_1(0) = p_2(0) = p_3(0) = 0$. Furthermore $V_{10}(0) = \langle \psi_1(0) | V | \psi_0(0) \rangle = 1$. Analogously, $V_{20}(0) = 0$, $V_{21}(0) = 2$, $V_{30}(0) = 0$, $V_{31}(0) = 0$, $V_{32}(0) = 1$. Integrating the dynamical system (3) with these initial data we find that $E_0(\lambda)$ and $E_1(\lambda)$ become smaller with increasing λ and $E_2(\lambda)$ and $E_3(\lambda)$ become larger with increasing λ. Furthermore the distances between adjacent levels become larger. This is called level repulsion. Notice that we have the constant of motion $E_0(\lambda) + E_1(\lambda) + E_2(\lambda) + E_3(\lambda) = C$ with $C = 10 = E_0(0) + E_1(0) + E_2(0) + E_3(0)$ or $10 = \text{Tr}\, H$, where Tr(.) denotes the trace. Obviously the problem described above can also be solved by calculating the eigenvalues directly for different λ's. The present method has the advantages that for all Hamiltonian operators we have the same equations of motion (3). Only the initial values must be changed for different systems.

In our second example we consider the Hamiltonian operator

$$H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{2}(x^2 + y^2) + \lambda x^2 y^2, \quad (6)$$

where $0 \leqq \lambda < \infty$. The spectrum of H is discrete and bounded from below. This system has also been studied by Pullen and Edmonds [5]. Obviously we put

$$H_0 = -\frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{2}(x^2 + y^2). \quad (7)$$

Thus H_0 describes two uncoupled harmonic oscillators. The underlying Hilbert space is $L_2(\mathbb{R}^2)$. The

normalized eigenfunctions for the operator H_0 are

$$u_{n_x n_y}(x, y) = \frac{1}{(\pi\, 2^{n_x+n_y}\, n_x!\, n_y!)^{1/2}} H_{n_x}(x) \cdot \exp(-x^2/2)\, H_{n_y}(y) \exp(-y^2/2)\,, \quad (8)$$

where $H_{n_x}(x)$ and $H_{n_y}(y)$ are Hermite polynomials. The full symmetry goup of the Hamiltonian H and H_0 is the C_{4v} point group. Then the basis functions we have chosen are linear combinations of the eigenfunctions given by (8) and transform according to the irreducible representation of C_{4v}. For each invariant subspace we have solved system (3) together with the initial data. Since we know the eigenfunctions and eigenvalues for the two uncoupled harmonic oscillators (7) we can easily find the initial data. Since the system has infinitely many energy levels we are forced to truncate the infinite system to a finite system. According to this truncation it is obvious that the higher lying energy levels are not accurate enough. Only the lower lying levels can be taken into account. Again we find level repulsion in the range $0 \leqq \lambda < \frac{1}{2}$. Our results coincide with those of Pullen and Edmonds [5]. They calculated the eigenvalues by diagonalizing truncated matrices of order 300.

Let us now discuss the connection with stationary state perturbation theory. The dynamical system (3) is an autonomous system of first order ordinary differential equations. Consequently the corresponding vector field is given by

$$S = \sum_{n=0}^{N} p_n \frac{\partial}{\partial E_n} + 2 \sum_{n=0}^{N} \left[\sum_{m(\neq n)}^{N} \frac{V_{nm} V_{mn}}{E_n - E_m} \right] \frac{\partial}{\partial p_n} + \sum_{\substack{m,n \\ m \neq n}}^{N} \left[\sum_{k(\neq m,n)}^{N} V_{mk} V_{kn} \left(\frac{1}{E_m - E_k} + \frac{1}{E_n - E_k} \right) - \frac{V_{mn}(p_m - p_n)}{E_m - E_n} \right] \frac{\partial}{\partial V_{mn}}, \quad (9)$$

where we have assumed for the time being that the system is finite (i.e. the number of eigenvalues is finite). $E_0(\lambda)$ denotes the smallest eigenvalue. From the theory of Lie series [6] we know that the general solution of the initial value problem is given by

$$\begin{pmatrix} E_0(\lambda) \\ \vdots \\ E_N(\lambda) \\ p_1(\lambda) \\ \vdots \\ p_N(\lambda) \\ V_{01}(\lambda) \\ \vdots \\ V_{N-1\,N}(\lambda) \end{pmatrix} = \exp(\lambda S) \begin{pmatrix} E_0 \\ \vdots \\ E_N \\ p_1 \\ \vdots \\ p_N \\ V_{01} \\ \vdots \\ V_{N-1\,N} \end{pmatrix} \Bigg| \begin{matrix} E_0 \to E_0(0) \\ \vdots \\ V_{N-1N} \to V_{N-1N}(0). \end{matrix} \quad (10)$$

Since $\exp(\lambda S) \equiv 1 + \lambda S + \lambda^2 S^2/2! + \ldots$ we find (1) to first order in λ. Including second order terms we obtain (2).

Let us now discuss the open questions. For an autonomous system of first order differential equations $\mathrm{d}x/\mathrm{d}\lambda = S(x)$ (S: analytic function, $x = (x_1, \ldots, x_N)^T$) the general solution of the initial value problem is given by $x(\lambda) = \exp(\lambda S)\, x|_{x \to x(0)}$ for sufficiently small λ. If the vector field S is linear then the Lie series solution holds for $0 \leqq \lambda < \infty$. On the other hand, if the system is nonlinear (such as system (3)) then the Lie series solution holds only for sufficiently small λ in general (even if the solution exists for $0 \leqq \lambda < \infty$). Then the Lie series technique must be combined with analytic continuation. It is an open question to estimate the convergence radius of the Lie series (10). The situation becomes even more complicated when the number of energy levels is infinite. This case appears when we study Bose systems. Thus we have to deal with infinite vector fields. Thus far we have assumed that (in the invariant subspace) the eigenvalues are non-degenerate. The question is: How must we modify system (3) in order to include accidental degeneracies? It can also happen that the eigenvalues of H_0 are non-degenerate. However with increasing coupling strength λ we obtain level crossing of certain adjacent levels. Such a case has been described by Steeb et al. [7]. Since system (1) includes terms of the form $(E_m - E_n)^{-1}$ on the right hand side it seems that we can integrate system (1) only in the range $0 \leqq \lambda < \lambda_c$ where λ_c denotes the crossing point. In many systems we have two or more external parameters. Let us consider the case with two external parameters. This means the Hamiltonian is of the form $H = H_0 + \lambda_1 V_1 + \lambda_2 V_2$. Let $\lambda = (\lambda_1, \lambda_2)$.

Defining

$$p_{kn}(\lambda) := \langle \psi_n(\lambda) | V_k | \psi_n(\lambda) \rangle \quad \text{and}$$

$$V_{kmn}(\lambda) := \langle \psi_m(\lambda) | V_k | \psi_n(\lambda) \rangle,$$

where $k = 1, 2$ we obtain the following system of partial differential equations

$$\frac{\partial E_n}{\partial \lambda_k} = p_{kn}, \quad \frac{\partial p_{kn}}{\partial \lambda_k} = 2 \sum_{m(\neq n)} \frac{V_{kmn} V_{knm}}{E_n - E_m},$$

$$\frac{\partial p_{1n}}{\partial \lambda_2} = \sum_{m(\neq n)} \frac{V_{1nm} V_{2mn} + V_{2nm} V_{1mn}}{E_n - E_m}, \tag{11}$$

$$\frac{\partial p_{2n}}{\partial \lambda_1} = \sum_{m(\neq n)} \frac{V_{2nm} V_{1mn} + V_{1nm} V_{2mn}}{E_n - E_m},$$

$$\frac{\partial V_{kmn}}{\partial \lambda_k} = \sum_{l(\neq m,n)} V_{kml} V_{kln} \left(\frac{1}{E_m - E_l} + \frac{1}{E_n - E_l} \right) + \frac{V_{kmn}(p_{km} - p_{kn})}{E_n - E_m},$$

$$\frac{\partial V_{1mn}}{\partial \lambda_2} = \sum_{l(\neq m,n)} (V_{2ml} V_{1ln} + V_{1ml} V_{2ln}) \cdot \left(\frac{1}{E_m - E_l} + \frac{1}{E_n - E_l} \right) + \frac{V_{2mn}(p_{1n} - p_{2m})}{E_m - E_n},$$

$$\frac{\partial V_{2mn}}{\partial \lambda_1} = \sum_{l(\neq m,n)} (V_{1ml} V_{2ln} + V_{2ml} V_{1ln}) \cdot \left(\frac{1}{E_m - E_l} + \frac{1}{E_n - E_l} \right) + \frac{V_{1mn}(p_{2n} - p_{1m})}{E_m - E_n},$$

where $k = 1, 2$. This system of partial differential equations can be solved with the help of ansatz (power series)

$$E_n(\lambda) = \sum_{j,k=0} E^n_{jk} \lambda_1^j \lambda_2^k, \quad p_n(\lambda) = \sum_{j,k=0} p^n_{jk} \lambda_1^j \lambda_2^k. \tag{12}$$

We find in second order of approximation

$$E_n(\lambda_1, \lambda_2) \approx E_n(0,0) + E^n_{10} \lambda_1 + E^n_{01} \lambda_2 + E^n_{20} \lambda_1^2 + E^n_{11} \lambda_1 \lambda_2 + E^n_{02} \lambda_2^2, \tag{13}$$

where

$$E^n_{10} = p^n_1(0,0), \quad E^n_{01} = p^n_2(0,0),$$

$$E^n_{20} = \sum_{m(\neq n)} \frac{V_{1mn}(0,0)\, V_{1mn}(0,0)}{E_n(0,0) - E_m(0,0)},$$

$$E^n_{02} = \sum_{m(\neq n)} \frac{V_{2mn}(0,0)\, V_{2mn}(0,0)}{E_n(0,0) - E_m(0,0)}, \tag{14}$$

$$E^n_{11} = \sum_{m(\neq n)} \frac{2\, V_{1mn}(0,0)\, V_{2mn}(0,0)}{E_n(0,0) - E_m(0,0)}.$$

The extension to n external parameters is straightforward. For two and more external parameters we must take into account the theorem of Wigner and von Neumann which was discovered in the very earliest days of quantum mechanics almost 60 years ago [8]. The theorem says that, among all self-adjoint Hermitian matrices, those with a degenerate eigenvalue have co-dimension three. When one has time-reversal invariance, as one does in the present case, one should be looking at real symmetric matrices, in which case the corresponding theorems says that they have co-dimension two. This is usually interpreted as saying that it requires at least two external parameters before one can expect any kind of eigenvalue crossing unless there is some symmetry reason that prevents the avoided crossings as the parameter is varied. This fact must be taken into account when we solve the system of partial differential equations given above.

Finally let us derive the energy dependent constants of motion for the Hamiltonian operator

$$H = \begin{pmatrix} a_{11} & \lambda a_{12} & \lambda a_{13} & \dots & \lambda a_{1N} \\ \lambda a_{12} & a_{22} & \lambda a_{23} & \dots & \lambda a_{2N} \\ \lambda a_{13} & \lambda a_{23} & a_{33} & \dots & \lambda a_{3N} \\ \vdots & & & & \\ \lambda a_{1N} & \lambda a_{2N} & \lambda a_{3N} & \dots & a_{NN} \end{pmatrix}, \tag{15}$$

where $a_{nm} \in \mathbb{R}$ $(n, m = 1, \dots, N)$ and λ is a real parameter.

Consequently H is a symmetric matrix. We assume further that the a_{mm}'s $(m = 1, \dots, N)$ are pairwise distinct. Obviously we put

$$H_0 = \operatorname{diag}(a_{11}, \dots, a_{NN}).$$

Therefore the eigenvalues of H_0 are not degenerate. The potential V is given by

$$V = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1N} \\ a_{12} & 0 & a_{23} & \dots & a_{2N} \\ a_{13} & a_{23} & 0 & \dots & a_{3N} \\ \vdots & & & & \\ a_{1N} & a_{2N} & a_{3N} & \cdots & 0 \end{pmatrix}. \tag{16}$$

In the following we give the energy dependent constants of motion for system (3), where the Hamiltonian is given by (15).

By inspection we find that

$$\sum_{m=1}^{N} p_m = \text{const} \tag{17}$$

is a constant of motion. Since $H_0 = \text{diag}(a_{11}, \ldots, a_{NN})$ we find that $E_1 = a_{11}, \ldots, E_N = a_{NN}$ are the eigenvalues of H_0. The corresponding eigenfunctions are the standard basis of $\mathbb{R}^N$, namely $|\psi_1(\lambda = 0)\rangle = (1, 0, \ldots, 0)^T$, $\ldots$, $|\psi_N(\lambda = 0)\rangle = (0, \ldots, 0, 1)^T$ (T means transpose). Consequently

$$p_m(\lambda = 0) = \langle \psi_m(\lambda = 0) \,|\, V \,|\, \psi_m(\lambda = 0) \rangle = 0 \tag{18}$$

for $m = 1, \ldots, N$. Since

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \sum_{m=1}^{N} E_m = \sum_{m=1}^{N} p_m = 0 \tag{19}$$

we find that $E_1 + E_2 + \ldots + E_N = \text{const}$ is a constant of motion. Consequently the initial data for system (3) are given by

$$E_m(\lambda = 0) = a_{mm}, \quad p_m(\lambda = 0) = 0$$

and

$$V_{mn}(\lambda = 0) = a_{mn} \quad (m < n).$$

Let us now derive the constants of motion which only depend on E. This derivation also includes explicitly λ-dependent constants of motion. The eigenvalues of H are determined by $\det(EI - H) = 0$, where I denotes the $N \times N$ unit matrix. Now this equation can be written as

$$\det(EI - H) \equiv E^N + h_{N-1} E^{N-1} + h_{N-2} E^{N-2} + \ldots + h_0 = 0, \tag{20}$$

where

$$\begin{aligned} h_{N-1} &= -\operatorname{tr}(H B_{N-1}) = -\operatorname{tr}(H) = -\sum_{m=1}^{N} a_{mm}, \\ h_{N-2} &= -\frac{1}{2} \operatorname{tr}(H B_{N-2}), \\ &\vdots \\ h_1 &= -\frac{1}{N-1} \operatorname{tr}(H B_1), \\ h_0 &= -\frac{1}{N} \operatorname{tr}(H B_0) \end{aligned} \tag{21}$$

with

$$\begin{aligned} B_{N-1} &= I, \\ B_{N-2} &= h_{N-1} I + H B_{N-1}, \\ B_{N-3} &= h_{N-2} I + H B_{N-2}, \\ &\vdots \\ B_1 &= h_2 I + H B_2, \\ B_0 &= h_1 I + H B_1. \end{aligned} \tag{22}$$

On the other hand we know that there exists an orthogonal matrix O such that $\tilde{H} = O^T H O$ with $\tilde{H} = \text{diag}(E_1, \ldots, E_N)$. From $\det(EI - \tilde{H}) = 0$ we obtain

$$\det(EI - \tilde{H}) \equiv E^N + \tilde{h}_{N-1} E^{N-1} + \tilde{h}_{N-2} R^{N-2} + \ldots + \tilde{h}_0 = 0, \tag{23}$$

where

$$\begin{aligned} \tilde{h}_{N-1} &= -\sum_{m=1}^{N} E_m, \\ \tilde{h}_{N-2} &= \sum_{m<n}^{N} E_m E_n, \\ &\vdots \\ \tilde{h}_0 &= (-1)^N E_1 E_2 \ldots E_N. \end{aligned} \tag{24}$$

From matrix theory we know that

$$\begin{aligned} \tilde{h}_{N-1} &= h_{N-1}, \\ \tilde{h}_{N-2} &= h_{N-2}, \\ &\vdots \\ \tilde{h}_0 &= h_0. \end{aligned} \tag{25}$$

From (25) we find the constant of motion given by (19). The coefficient h_{N-1} does not depend on λ. However each of the other coefficients $h_{N-2}, \ldots, h_0$ may depend on λ. The coefficients $h_{N-2}, \ldots, h_0$, which are polynomials in λ, can be written as $h_m = h_m^c + h_m^\lambda$, where h_m^c is the λ independent part and h_m^λ is the λ dependent part. Consequently we find the constants of motion

$$\tilde{h}_m - h_m^\lambda = h_m^c \tag{26}$$

with $m = N - 2, \ldots, 0$.

As an example, let

$$H = \begin{pmatrix} 1 & \lambda & 0 & 0 \\ \lambda & 2 & 2\lambda & 0 \\ 0 & 2\lambda & 3 & \lambda \\ 0 & 0 & \lambda & 4 \end{pmatrix}. \tag{27}$$

Then the eigenvalues equation $\det(EI - H) = 0$ yields

$$E^4 - 10E^3 + (35 - 6\lambda^2) E^2 + (-50 + 30\lambda^2) E + 24 - 20\lambda^2 + \lambda^4 = 0. \tag{28}$$

Consequently we obtain the constants of motion

$$\begin{aligned} \sum_{m=1}^{4} E_m(\lambda) &= 10, \\ \sum_{m<n}^{4} E_n(\lambda) E_m(\lambda) + 6\lambda^2 &= 35, \\ \sum_{n<m<k}^{4} E_n(\lambda) E_m(\lambda) E_k(\lambda) + 30\lambda^2 &= 50, \\ E_1(\lambda) E_2(\lambda) E_3(\lambda) E_4(\lambda) + 30\lambda^2 - \lambda^4 &= 24. \end{aligned} \tag{29}$$

[1] P. Pechukas, Phys. Rev. Lett. **51,** 943 (1983).
[2] T. Yukawa, Phys. Rev. Lett. **17,** 1883 (1985).
[3] T. Yukawa, Phys. Lett. A **116,** 227 (1986).
[4] W.-H. Steeb and J. A. Louw, Chaos and Quantum Chaos, World Scientific, Singapore 1986.
[5] R. A. Pullen and A. R. Edmonds, J. Phys. A: Math. Gen. **14,** L 447, (1981).
[6] W. Gröbner and H. Knapp, Contributions to the Method of Lie Series, Bibliographisches Institut, Mannheim 1967.
[7] W.-H. Steeb, J. A. Louw, C. M. Villet, and A. Kunick, Physica Scripta **34,** 245 (1986).
[8] J. v. Neumann and E. Wigner, Physik. Zeitschr. **30,** 467 (1929).